Convertible, Nearly Decomposable, and Nearly Reducible Matrices

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ABSTRACT

We use basic facts about convertible matrices, nearly decomposable matrices, and nearly reducible matrices to give a matrix theoretical proof of the known characterizations of convertible matrices.

1. INTRODUCTION

Let $A = [a_{ij}]$ be a (0, 1) matrix of order n. The permanent of A and the determinant of A may be defined by

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \tag{1}$$

and

$$\det A = \sum_{\sigma} (\operatorname{sgn} \sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \tag{2}$$

respectively, where the summation is over all permutations σ of $\{1, \ldots, n\}$, and sgn σ denotes the sign of the permutation σ. Despite its resemblance to

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the determinant, the permanent has proven to be more difficult to compute (see [8]). However, for matrices with certain zero-nonzero patterns it is possible to evaluate the permanent by calculating the determinant of a closely related matrix. For example, let $a, \dots, m$ be real numbers, and let V and W be the matrices of order 4 defined by

$$V = \begin{bmatrix} 0 & a & b & c \\ d & 0 & e & f \\ g & h & 0 & i \\ k & l & m & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & a & b & c \\ d & 0 & -e & f \\ -g & -h & 0 & i \\ k & -l & m & 0 \end{bmatrix}.$$

It is easy to verify that each term in the expansion of $\operatorname{per} V$ equals the corresponding term in the expansion of $\det W$. Hence, we conclude that $\operatorname{per} V = \det W$ for all real numbers $a, \dots, m$. The notion of evaluating a permanent via a determinant and the following concepts are discussed in [5] and [6].

Assume $\operatorname{per} A \neq 0$, and let $B = [b_{ij}]$ be a $(0, 1, -1)$ matrix of order n such that

$$|b_{ij}| = a_{ij} \qquad (1 \leqslant i, j \leqslant n). \tag{3}$$

Let $\mathscr{M}(A)$ denote the set of all real matrices of order n which have 0's in those positions in which A has 0's (and possibly elsewhere). Then the matrix B *converts* the permanent of matrices in $\mathscr{M}(A)$ into their determinant provided

$$\operatorname{per} M = |\det B*M| \qquad \text{for all} \quad M \in \mathscr{M}(A),$$

where $*$ denotes the Hadamard product. In this case, B is a *conversion* of A, and A is a *convertible* matrix. It is well known that B is a conversion of A if and only if either every nonzero term in the determinantal expansion of B is positive or every nonzero term is negative. Thus B converts the permanent of matrices in $\mathscr{M}(A)$ into the determinant if and only if (3) holds and $\operatorname{per} A = |\det B|$. The square $(0, 1, -1)$ matrix B is a *sign-nonsingular matrix*, abbreviated SNS matrix, provided each real matrix with the same sign pattern as B is nonsingular (that is, provided $B*M$ is invertible for every matrix M each of whose entries is positive). In [2] it is observed that B is a conversion of A if and only if (3) holds and B is an SNS matrix.

The notions of convertible matrices and conversions can be reformulated in terms of directed graphs as follows. Let $\mathscr{D}$ be a digraph with vertex set V and arc set E. A *weighting of the digraph* $\mathscr{D}$ is a labeling, $\mathbf{wt}: E \to \{0, 1\}$, of its arcs with 0's and 1's. The pair $(\mathscr{D}, \mathbf{wt})$ is a *weighted digraph*. The *weight* of a path or a directed cycle γ of $(\mathscr{D}, \mathbf{wt})$ is denoted by $\mathbf{wt}\,\gamma$ and equals the sum of the weights of its arcs modulo 2. The digraph $\mathscr{D}$ is an *even digraph* provided that for each weighting $\mathbf{wt} : E \to \{0, 1\}$ of $\mathscr{D}$ there exists a directed cycle γ (depending on $\mathbf{wt}$) of $\mathscr{D}$ such that $\mathbf{wt}\,\gamma \equiv 0 \bmod 2$. Thus $\mathscr{D}$ is not an even digraph if and only if there exists a weighting $\mathbf{wt}$ of $\mathscr{D}$ such that $\mathbf{wt}\,\gamma \equiv 1 \bmod 2$ for every directed cycle γ of $\mathscr{D}$.

Let $B = [b_{ij}]$ be $(0, 1, -1)$ matrix of order n. The *digraph of the matrix* B is the digraph with vertices $1, \ldots, n$ and with an arc (i, j) from i to j if and only if $i \neq j$ and $b_{ij} \neq 0$. The *weighted digraph* of the matrix B is $(\mathscr{D}(B), \mathbf{wt})$ where $\mathbf{wt}(i, j) = 0$ if $b_{ij} = 1$ and $\mathbf{wt}(i, j) = 1$ if $b_{i,j} = -1$. The following characterization of SNS matrices in terms of weighted digraphs is due to Bassett, Maybee, and Quirk [1].

THEOREM 1. *Let $B = [b_{ij}]$ be a $(0, 1, -1)$ matrix of order n with $b_{ii} = -1$ $(i = 1, \ldots, n)$, and let $(\mathscr{D}(B), \mathbf{wt})$ be the weighted digraph of B. Then B is an SNS matrix if and only if $\mathbf{wt}\,\gamma \equiv 1 \bmod 2$ for every directed cycle γ of $\mathscr{D}(B)$.*

If the $(0, 1)$ matrix A is convertible and $\hat{A}$ is a conversion of A, then any matrix obtained from $\hat{A}$ by multiplying some of its rows and some of its columns by -1 is also a conversion of A. As an immediate consequence of Theorem 1 and the relationship between SNS matrices and conversions, we have the following characterization of even digraphs.

COROLLARY 2. *Let $A = [a_{ij}]$ be a $(0, 1)$ matrix of order n with $a_{ii} = 1$ $(i = 1, \ldots, n)$, and let $\mathscr{D}(A)$ be the digraph determined by A. Then A is convertible if and only if $\mathscr{D}(A)$ is not an even digraph.*

In light of Corollary 2, we say that the digraph $\mathscr{D}$ is *convertible* if it is not even, and we call a weighting for which the weight of every directed cycle of $\mathscr{D}$ equals 1 mod 2 a *conversion* of the digraph $\mathscr{D}$.

Corollary 2 shows that the problem of determining whether or not a matrix is convertible is equivalent to the problem of deciding whether or not a digraph is even. At this time the complexity of either problem is unknown. Little [6] reduces the problem of testing a matrix A for convertibility to that of searching for the presence of certain matrices among the submatrices of A. Seymour and Thomassen [7] show that testing a digraph for evenness is equivalent to determining if the digraph contains certain subdigraphs. We now discuss these reductions in more detail.

Let $A = [a_{ij}]$ be a $(0, 1)$ matrix of order n. Let i and j be integers such that $a_{ij} = 1$, and suppose that row i contains exactly two 1's. There exists a unique integer $k \neq j$ such that $a_{ik} = 1$. Let C be the $(0, 1)$ matrix of order

$n - 1$ obtained from A by replacing column k with the column whose entry in row l is 1 if and only if either $a_{lj} = 1$ or $a_{lk} = 1$, and then deleting row i and column j of A. Then C is the matrix obtained from A by the *row contraction* on the 1 in position (i, j). The notion of the *column contraction* on a 1 is defined analogously. It is not difficult to show that a row or column contraction of A is convertible if and only if A is convertible. Let M be a $(0, 1)$ matrix of order $m \leqslant n$. The matrix A is *contractible* to M provided $m = n$ and $M = A$, or there is a sequence of matrices $A = A_0, A_1, \dots, A_{n-m} = M$ such that A_{i+1} is a row or column contraction of A_i $(i = 0, \dots, n - m - 1)$. Let J_3 denote the all 1's matrix of order 3, and I an identity matrix (which is possibly vacuous). If E and F are real matrices of the same size, we write $F \leqslant E$ provided that each element of $E - F$ is nonnegative. The following characterization of convertible $(0, 1)$ matrices is a consequence of Corollary 1 of [6] (see also [5, Theorem 1.1]).

THEOREM 3. *Let A be a $(0, 1)$ matrix of order n such that* per $A \neq 0$. *Then A is convertible if and only if there does not exist a $(0, 1)$ matrix $C \leqslant A$ whose rows and columns can be permuted to obtain a matrix of the form*

$$I \oplus Y, \tag{4}$$

where Y is contractible to J_3.

Theorem 3 characterizes convertible matrices as those matrices which do not contain any submatrix of a certain specified type. Let $A = [a_{ij}]$ be a $(0, 1)$ matrix of order n with per $A \neq 0$. Since the convertibility of a matrix is preserved under row and column permutations, we may assume without loss of generality that $A = [a_{ij}]$ has 1's on its main diagonal. We now describe the Seymour and Thomassen characterization, which, in contrast to Theorem 3, is in terms of the digraph of A.

Let (x, y) be an arc of a digraph $\mathscr{D}$. By *splitting the arc* (x, y) we mean adjoining a new vertex z to $\mathscr{D}$ and replacing the arc (x, y) with the arcs (x, z) and (z, y). A *subdivision* of $\mathscr{D}$ is a digraph obtained from $\mathscr{D}$ by successively splitting arcs. We consider the digraph as a subdivision of itself. Let x be a vertex of $\mathscr{D}$. By *splitting the vertex* x we mean adjoining a new vertex z and a new arc (x, z) and then replacing each arc of the form (x, y), $y \neq z$, with the arc (z, y). A *splitting* of the digraph $\mathscr{D}$ is a subdivision of a digraph obtained from $\mathscr{D}$ by splitting some of its vertices. The digraph $\mathscr{D}$ is considered a splitting of itself.

Let G be a graph. We denote by G^* the digraph obtained from G by replacing each edge $\{u, v\}$ with the oppositely directed arcs (u, v) and (v, u). If G is a directed cycle of length k, then G^* is called a *k-double cycle*. An *odd-double cycle* is a k-double cycle for some odd integer $k \geqslant 3$. The digraph G^* is a *double tree* if G is a tree, and is a *double path* if G is a path. The following theorem is the Seymour and Thomassen [7] characterization of even digraphs.

THEOREM 4. *The digraph $\mathscr{D}$ is an even digraph if and only if $\mathscr{D}$ contains a splitting of an odd-double cycle.*

Using Corollary 2 we obtain a characterization of convertible matrices [5, Corollary 1.3].

THEOREM 5. *Let A be a $(0,1)$ matrix of order n all of whose main diagonal elements equal 1. Then A is convertible if and only if $\mathscr{D}(A)$ does not contain a splitting of an odd-double cycle.*

Let A be a $(0,1)$ matrix of order n each of whose main diagonal entries equals 1. Then, as discussed in [5], one can show directly that $\mathscr{D}(A)$ contains a splitting of an odd-double cycle if and only only if there exists a $(0,1)$ matrix $C \leqslant A$ whose rows and columns can be permuted to obtain a matrix of the form $I \oplus Y$, where Y is contractible to J_3. Thus the characterizations of convertible matrices in Theorems 3 and 5 are equivalent. Indeed, as we shall see, Little's proof, and Seymour and Thomassen's proof share a common kernel. Before discussing the structure of their proofs, we recall a few basic definitions and results.

Let A be a $(0,1)$ matrix of order $n \geqslant 2$. Then the matrix A is *partly decomposable* if there exist permutation matrices P and Q of order n such that PAQ has the form

$$\left[\begin{array}{c|c} B_1 & O \\ \hline X & B_2 \end{array}\right], \tag{5}$$

where B_1 and B_2 are square (nonvacuous) matrices. If A is not partly decomposable, then A is *fully indecomposable*. It is easy to verify that a row or column contraction of A is fully indecomposable if and only if A is fully indecomposable. If A is fully indecomposable and each matrix obtained from A by replacing one of its 1's with a 0 is partly decomposable, then A is *nearly decomposable*. The matrix A is *reducible* provided there exists a permutation matrix P such that PAP^T has the form (5), and is *irreducible* if no such P exists. If A is irreducible and each matrix obtained from A by replacing one of its 1's by a 0 is reducible, then A is *nearly reducible*. A digraph $\mathscr{D}$ is *strongly connected* if for each pair of distinct vertices v and w there exists a path from v to w. The digraph $\mathscr{D}$ is a *minimal strong digraph* provided it is strongly connected and removing any arc of $\mathscr{D}$ results in a digraph that is not strongly connected. It is well known that A is irreducible if and only if $\mathscr{D}(A)$ is strongly connected, and that if each diagonal entry of A is nonzero then A is fully indecomposable if and only if A is irreducible. Also, A is nearly reducible if and only if $\mathscr{D}(A)$ is a minimal strong digraph.

Let $A = [a_{ij}]$ be a (0, 1) matrix of order $n \geqslant 2$ with each diagonal entry equal to 1, and assume that A is fully indecomposable, or equivalently A is irreducible. Let $X_1 = [1]$, and for $l \geqslant 2$ let X_l denote the (0, 1) matrix of order l with 1's on the main diagonal, 1's in positions $(1, 2), \dots, (l-1, l)$, and 0's elsewhere. It is easy to see that there exists a nonnegative integer k and a sequence of matrices $A_0, \dots, A_k$ such that

(i) for some $l_0 \geqslant 2$ and some permutation matrix P_0 we have that $P_0^T A_0 P_0$ is the (0, 1) matrix of order l_0 with 1's on its main diagonal, 1's in positions $(1, 2), \dots, (l_0 - 1, l_0)$, and $(l_0, 1)$, and 0's elsewhere;

(ii) if $1 \leqslant i \leqslant k$, then either A_i is obtained from A_{i-1} by replacing a single 0 entry with a 1, or for some integer l_i and some permutation matrix P_i we have

$$P_i^T A_i P_i = \begin{bmatrix} X_{l_i} & E_1 \\ E_2 & A_{i-1} \end{bmatrix},$$

where E_2 has a 1 in its first row and first column and 0's elsewhere, and E_1 has a 1 in its last row and first column and 0's elsewhere;

(iii) $A = A_k$.

Each of the matrices A_i $(0 \leqslant i \leqslant k)$ is a fully indecomposable matrix with 1's on its main diagonal and hence is an irreducible matrix. It is clear that each A_i determines a set of indices α_i such that $\alpha_{i-1} \subseteq \alpha_i$ and the principal submatrix $A[\alpha_i, \alpha_i]$ of A determined by the rows and columns of A with index in α_i satisfies $A_i \leqslant A[\alpha_i, \alpha_i]$. Let $\overline{A}_i$ be the (0, 1) matrix of order n with principal submatrix in rows and columns indexed by α_i equal to A_i, 1's on its main diagonal, and 0's elsewhere. Then $\overline{A}_{i-1} \leqslant \overline{A}_i$ for $1 \leqslant i \leqslant k$.

There is an analogous decomposition for digraphs. Let $\mathscr{D}$ be a strongly connected digraph. Then there exists a positive integer l and a sequence of subdigraphs $\mathscr{D}_1, \dots, \mathscr{D}_l$ of $\mathscr{D}$ such that

(i′) $\mathscr{D}_1$ is a directed cycle;

(ii′) if $1 \leqslant i \leqslant l$, then $\mathscr{D}_i$ is obtained from $\mathscr{D}_{i-1}$ by either adding a new arc between two vertices of $\mathscr{D}_{i-1}$ or appending the vertices and the arcs of a path in $\mathscr{D}$ whose initial and terminal vertices are in $\mathscr{D}_{i-1}$ and whose interior vertices are not vertices of $\mathscr{D}_{i-1}$; and

(iii′) $\mathscr{D} = \mathscr{D}_l$.

Suppose E and F are (0, 1) matrices of order n with $E \leqslant F$, and assume $\tilde{E}$ is a conversion of E. Then a conversion $\tilde{F}$ of F is an *extension* of $\tilde{E}$ provided $\tilde{E}*\tilde{F} = E$. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are digraphs such that every arc of $\mathscr{D}_1$ is an arc of $\mathscr{D}_2$, then a weighting $\mathbf{wt}_2$ of $\mathscr{D}_2$ is an *extension* of the weighting $\mathbf{wt}_1$ of $\mathscr{D}_1$ provided $\mathbf{wt}_2$ and $\mathbf{wt}_1$ agree on the arcs of $\mathscr{D}_1$.

Let A be a (0, 1) matrix of order n such that each diagonal entry of A equals 1. Suppose that $A_1, A_2, \ldots, A_k$ are matrices satisfying (i), (ii), and (iii). Little's proof of Theorem 3 is based on establishing the validity of the following algorithm.

ALGORITHM L.

Let $\tilde{A}_0$ be a conversion of $\bar{A}_0$.
Set Flag to true and set i to 0.
While (Flag is true and $i < k$) do
  If the conversion $\tilde{A}_i$ can be extended to a conversion of $\bar{A}_{i+1}$, then let A be one such extension and increment i by 1.
  Otherwise, set Flag to false.
If Flag = true then A is convertible and A_k is a conversion of A.
If Flag = false then A is not convertible and there exists a matrix $C \leqslant A$ which can be permuted to the form $I \oplus Y$ where Y is contractible to J_3.

The proof of Theorem 4 presented by Seymour and Thomassen is based on establishing the validity of the following algorithm, which is essentially the same as the previous algorithm. Suppose that $\mathscr{D}_1, \ldots, \mathscr{D}_l$ are digraphs satisfying (i′), (ii′), and (iii′), where $\mathscr{D}_l = \mathscr{D}(A)$.

ALGORITHM ST.

Let $\mathbf{wt}_0$ be a conversion of $\mathscr{D}_0$.
Set Flag to true and set i to 0.
While (Flag is true and $i < l$) do
  If the conversion $\mathbf{wt}_i$ can be extended to a conversion of $\mathscr{D}_{i+1}$, then let $\mathbf{wt}_{i+1}$ be one such extension and increment i by 1.
  Otherwise, set Flag to false.
If Flag = true then $\mathscr{D}(A)$ is not an even digraph and $\mathbf{wt}_l$ is a conversion of $\mathscr{D}(A)$.
If Flag = false then $\mathscr{D}(A)$ is an even digraph and $\mathscr{D}(A)$ contains a splitting of an odd-double cycle.

Both proofs involve somewhat lengthy arguments to establish the validity of the final statement of the algorithm. In general the sequences $A_0, \ldots,$

$A_k = A$ and $\mathscr{D}_0, \dots, \mathscr{D}_l = \mathscr{D}(A)$ are not unique. By a judicious choice of the sequences we are able to prove Theorems 3 and 4 in a unified manner, using only basic results on convertible, nearly decomposable, and nearly reducible matrices. It is hoped that the new proofs will help illuminate the relationship between the two approaches. At the very heart of the proofs we encounter nearly decomposable matrices and nearly reducible (0, 1) matrices of order n with the largest number of 1's. This establishes a beautiful relationship between nearly reducible matrices and nearly decomposable matrices.

We begin with a few preliminary results in Section 2 and then present new proofs of Theorems 3 and 4 in Sections 3 and 4, respectively.

2. PRELIMINARIES

Let $A = [a_{ij}]$ be a (0, 1) matrix of order $n \geqslant 2$. As is customary, we let $A(i|j)$ denote the matrix of order $n - 1$ obtained from A by deleting row i and column j, and let E_{ij} denote the matrix of order n whose unique nonzero entry is a 1 in position (i, j). It is well known that A is fully indecomposable if and only if per $A(i|j) \neq 0$ $(1 \leqslant i, j \leqslant n)$. We begin this section by recalling two results about nearly decomposable and nearly reducible (0, 1) matrices (see [4, Theorems 4.3.5 and 3.3.5]).

PROPOSITION 6. *A nearly decomposable* (0, 1) *matrix of order $n \geqslant 3$ has at most $3n - 3$ ones.*

PROPOSITION 7. *Let A be a nearly reducible matrix of order n. Then A has at most $2n - 2$ nonzero entries, with equality if and only if each diagonal entry of A equals 0 and $\mathscr{D}(A)$ is a double tree with n vertices.*

Proposition 6 implies that if $A = [a_{ij}]$ is a (0, 1) matrix of order n such that A has exactly three 1's in its first column, exactly three 1's in its first row, and at least three 1's in each of the remaining rows and columns, $a_{11} = 1$, and $A[1|1]$ is nearly decomposable, then $n = 3$ and A is the matrix of all 1's. Proposition 7 implies that if $A = [a_{ij}]$ is a (0, 1) matrix of order n such that $a_{ii} = 0$ $(i = 1, 2, \dots, n)$, the first column of A has exactly two 1's, the first row of A has exactly two 1's, each of the remaining rows and columns have at least two 1's, and $A[1|1]$ is nearly reducible, then $\mathscr{D}(A)$ is an n-double cycle. Thus the fundamental forbidden configurations given in Theorems 3 and 4 are related to nearly decomposable and nearly reducible matrices with the largest number of 1's, respectively. These simple facts are used at crucial stages of our proofs of Theorems 3 and 4 and, as we shall see, encapsulate the essential difference between the two characterizations of convertible matrices.

We shall need the following elementary fact about strongly connected digraphs. Let $\mathscr{D}$ be a digraph with vertex set X and arc set E. If S is a subset of V, then we denote by $\mathscr{D}_S$ the subdigraph of $\mathscr{D}$ whose vertex set is $V \setminus S$ and arc set consists of those arcs in E between two vertices in $V \setminus S$.

PROPOSITION 8. *Let $\mathscr{D}$ be a strongly connected (loopless) digraph such that each vertex of $\mathscr{D}$ has outdegree at least 2. Then there exists a vertex v such that $\mathscr{D}_{\{v\}}$ is a strongly connected digraph.*

Proof. Let V be the vertex set of $\mathscr{D}$, and choose a nonempty subset S of V of minimal cardinality such that $\mathscr{D}_S$ is strongly connected. Let $T = V \setminus S$. Since $\mathscr{D}$ is strongly connected, there exists an arc from a vertex u in T to a vertex v in S and a path γ from v to a vertex u' in T such that the internal vertices of γ are in S. The choice of S implies that each vertex of S lies on γ, and there is at most one vertex w in S such that (v, w) is an arc of $\mathscr{D}$. Since v has outdegree at least 2, there exists a vertex u^* in T such that (v, u^*) is an arc of $\mathscr{D}$. Thus $\mathscr{D}_{S \setminus \{v\}}$ is strongly connected, and by the minimality of S we must have $S = \{v\}$. ■

An immediate consequence of the previous proposition is the following result of [3].

PROPOSITION 9. *Let $A = [a_{ij}]$ be a fully indecomposable $(0, 1)$ matrix of order n with at least three 1's in each column. Then there exist integers r and s such that $a_{rs} = 1$ and $A(r|s)$ is fully indecomposable.*

We now recall a few theorems about the set of conversions of a convertible matrix. The next two theorems are restatements of Theorems 2.1 and 2.2 of [5]. The first implies that there is essentially only one conversion of a fully indecomposable convertible matrix, and the second concerns conversions and their extensions.

THEOREM 10. *Let A be a fully indecomposable $(0, 1)$ matrix of order n. Suppose that A is convertible and that B_1 and B_2 are conversions of A. Then there exist diagonal matrices D_1 and D_2 such that each of their main diagonal entries is ± 1 and $D_1 B_1 D_2 = B_2$.*

THEOREM 11. *Let $A = [a_{ij}]$ be a fully indecomposable $(0, 1)$ matrix of order n, and let B be a conversion of A. Suppose that $a_{rs} = 0$. Then the following are equivalent:*

(i) *$A + E_{rs}$ is convertible.*
(ii) *There exists a conversion of $A + E_{rs}$ which is an extension of B.*
(iii) *The matrix $B(r|s)$ is a sign-nonsingular matrix.*

Let $(\mathscr{D}, \mathbf{wt})$ be a weighted digraph with arc set E. For a subset S of vertices of $\mathscr{D}$, we define $\mathbf{wt}^S : E \to \{0, 1\}$ by

$$\mathbf{wt}^S\ \alpha \equiv \begin{cases} \mathbf{wt}(\alpha) + 1 \pmod 2 & \text{if } \alpha \text{ is incident to exactly one vertex in } S, \\ \mathbf{wt}(\alpha) \pmod 2 & \text{otherwise.} \end{cases}$$

Clearly $\mathbf{wt}\ \gamma \equiv \mathbf{wt}^S\ \gamma \bmod 2$ for any directed cycle γ of $\mathscr{D}$. We now restate Theorems 10 and 11 in terms of weighted digraphs.

THEOREM 12. *Let $\mathscr{D}$ be a strongly connected digraph, and suppose that $\mathbf{wt}_1$ and $\mathbf{wt}_2$ are conversions of $\mathscr{D}$. Then there exists a subset S of vertices such that $\mathbf{wt}_1$ equals $\mathbf{wt}_2^S$.*

THEOREM 13. *Let $\mathscr{D}$ be a strongly connected digraph and $\mathbf{wt}$ a conversion of $\mathscr{D}$. Suppose that there is not an arc from vertex r to vertex s and that $r \neq s$. Let $\mathscr{D}'$ be the digraph obtained from $\mathscr{D}$ by including the arc (r, s). Then the following are equivalent:*

(i) *$\mathscr{D}'$ is convertible,*
(ii) *there exists a conversion of $\mathscr{D}'$ which is an extension of $\mathbf{wt}$,*
(iii) *if α and β are paths from s to r in $\mathscr{D}$, then $\mathbf{wt}\ \alpha \equiv \mathbf{wt}\ \beta \bmod 2$.*

3. LITTLE'S CHARACTERIZATION

In this section we provide a new proof of Theorem 3. We begin with a technical lemma.

LEMMA 14. *Let $A = [a_{ij}]$ be a $(0, 1)$ matrix of order $n \geqslant 3$. Suppose that $a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = 1$ and that the remaining entries in the first row and column of A are zero. Assume that $A(1, 1)$ is a fully indecomposable, convertible matrix and that B is a conversion of $A(1, 1)$. Then A is convertible if and only if each of $B(1|1)$, $B(1|2)$, $B(2|1)$, and $B(2|2)$ is an SNS matrix and*

$$\det B(1|1) \det B(1|2) \det B(2|1) \det B(2|2) > 0. \tag{6}$$

Proof. The assumptions on A imply that each of the matrices $A - E_{12}$, $A - E_{13}$, $A - E_{21}$, and $A - E_{31}$ is a fully indecomposable matrix.

First suppose that A is convertible. Then the matrix $A - E_{13}$ is a convertible matrix. It follows from the full indecomposability of $A(1|1)$ and Theorems 10 and 11 that there exists a conversion $\tilde{A} = [\tilde{a}_{ij}]$ of A such that

$\tilde{A}[1|1] = B$. We have

$$\det \tilde{A}(1|2) = \tilde{a}_{21} \det B(1|1) - \tilde{a}_{31} \det B(2|1)$$

and

$$\det \tilde{A}(1|3) = \tilde{a}_{21} \det B(1|2) - \tilde{a}_{31} \det B(2|2).$$

Since $\tilde{A}$ is an SNS matrix, the nonzero terms in the expansion of $\det \tilde{A}$ have the same sign. The full indecomposability of B implies that each of $B(1|1)$, $B(1|2)$, $B(2|1)$, and $B(2|2)$ is an SNS matrix. It follows that $\tilde{a}_{21} \det B(1|1)$ and $-\tilde{a}_{31} \det B(2|1)$ have the same sign, and hence

$$-\tilde{a}_{21}\tilde{a}_{31} \det B(1|1) \det B(2|1) > 0.$$

Similarly,

$$-\tilde{a}_{21}\tilde{a}_{31} \det B(1|2) \det B(2|2) > 0.$$

We conclude that

$$\det B(1|1) \det B(1|2) \det B(2|1) \det B(2|2) > 0$$

and hence the "only if" implication of the lemma is proven.

Now suppose that each of $B(1|1)$, $B(1|2)$, $B(2|1)$, and $B(2|2)$ is an SNS matrix and that their determinants satisfy (6). Let D_1 and D_2 be diagonal matrices of order $n-1$ each of whose main diagonal entries equals ± 1. Then $D_1 B D_2$ is an SNS matrix and

$$\det D_1 BD_2(1|1) \det D_1 BD_2(1|2) \det D_1 BD_2(2|1) \det D_1 BD_2(2|2) > 0.$$

Thus without loss of generality we may assume that

$$\det B(i|j) > 0 \qquad \text{for} \quad 1 \leqslant i, j \leqslant 2. \tag{7}$$

Let x equal ± 1 according as $\det B$ is positive or negative. Let $\tilde{A} = [\tilde{a}_{ij}]$ be the $(0, 1, -1)$ matrix of order n with the same zero pattern as A such that

$$\tilde{A}(1|1) = B \qquad \tilde{a}_{11} = 1, \quad \tilde{a}_{21} = 1, \quad \tilde{a}_{31} = -1, \quad \tilde{a}_{12} = -x, \text{ and } \tilde{a}_{13} = x.$$

By Laplace expansion along the first row of $\tilde{A}$ we obtain

$$\det \tilde{A} = \det B + x[\det B(1|1) + \det B(2|1)] + x[\det B(1|2) + \det B(2|2)].$$

Since (7) holds and the matrices B and $B(i|j)$ $(1 \leqslant i, j \leqslant 2)$ are SNS matrices, it follows that each nonzero term in the expansion of the determinant of $\tilde{A}$ has the same sign. Hence A is convertible. ■

We now present our new proof of Theorem 3. First we present an outline for the proof that a (0, 1) matrix $A = [a_{ij}]$ of order n which has nonzero permanent and is not convertible must contain a forbidden configuration as given in the statement of Theorem 3. The proof is by induction on n.

(1) We may assume that A is fully indecomposable and that A is minimal in the sense that replacing any 1 of A with a 0 results in a matrix that is either partly decomposable or is convertible.

(2) If A has a row or a column with exactly two 1's, then we can apply induction to a contraction of A.

(3) Assume that each row and column of A has at least three 1's. By Proposition 9, there exist integers i and j such that $a_{ij} = 1$ and $A[i|j]$ is fully indecomposable.

(4) Use the minimality of A to conclude that row i and column j of A have exactly three 1's each.

(5) Use Lemma 14 and the minimality of A to conclude that $A[i|j]$ is nearly decomposable.

(6) Use Proposition 6 to conclude that $n = 3$ and $A = J_3$.

Proof of Theorem 3. It is easy to verify that J_3 is not convertible. Hence any matrix of the form $I \oplus Y$ where Y is contractible to J_3 is not convertible. We now show that any matrix which is not convertible contains a matrix of this type. Our proof will be by induction on the order of the matrix.

Let A be a (0, 1) matrix of order $n \geqslant 2$ with per $A \neq 0$, and assume that A is not convertible. Since the all 1's matrix of order 2 is convertible, $n \geqslant 3$. If A is partly decomposable, then we may apply induction to the fully indecomposable components of A. Thus, we assume that A is fully indecomposable and that A is minimal in the sense that replacing any 1 of A with 0 results in a matrix that is either partly decomposable or convertible.

Suppose A has a row or column with exactly two 1's. Then A can be contracted to a matrix B of order $n - 1$. Since contraction preserves convertibility and nonconvertibility, B is not convertible. Contraction also preserves fully indecomposability. Hence by induction, there exists a $C \leqslant B$ of the form (4). It follows that there exists a $C' \leqslant A$ of the form (4).

Thus we may assume that each row and column of A has at least three 1's. By Proposition 9, there exist integers r and s such that $a_{rs} = 1$ and $A(r|s)$ is fully indecomposable. Without loss, assume that $r = s = 1$. Suppose there are at least four 1's in the first row of A and that $a_{12} = a_{13} = a_{14} = 1$. Then each of the matrices $A - E_{12}$, $A - E_{13}$, and $A - E_{12} - E_{13}$ is fully indecomposable. Hence, by the minimality of A, each is convertible. Let $\tilde{C}$ be a conversion of $A - E_{12} - E_{13}$. Since $A - E_{12}$ is convertible, Theorem 11 implies that $\tilde{C}(1|2)$ is an SNS matrix. A similar argument shows that $\tilde{C}(1|3)$ is an SNS matrix. Let $\tilde{\tilde{C}}$ be the matrix obtained from $\tilde{C}$ by replacing the 0 in position (1, 3) with ± 1 according to the sign of det $\tilde{C}$ det $\tilde{C}(1|3)$. Then $\tilde{\tilde{C}}$ is an SNS matrix. Since $\tilde{\tilde{C}}(1|2) = \tilde{C}(1|2)$, the matrix $\tilde{\tilde{C}}(1|2)$ is an SNS matrix. Theorem 11 now implies that A is convertible, contrary to our assumption. We conclude that row 1, and similarly column 1, of A have exactly three 1's.

Let $B = A(1|1)$. First suppose that B is not convertible. Then, by induction, there exists a matrix $C \leqslant B$ of the form (4). It follows that the matrix $[1] \oplus C$ has the form (4) and $[1] \oplus C \leqslant B$. Now suppose that B is convertible, and let $\tilde{B}$ be a conversion of B. The matrix $A - E_{13}$ is fully indecomposable, and hence, by the minimality of A, $A - E_{13}$ is convertible. The full indecomposability of B and Theorems 10 and 11 imply that there exists a conversion $\widetilde{A - E_{13}}$ of $A - E_{13}$ with $\widetilde{A - E_{13}}(1|1) = \tilde{B}$. It now follows that $\tilde{B}(1|1)$ and $\tilde{B}(2|1)$ are both SNS matrices. A similar argument applied to $A - E_{12}$ implies that $\tilde{B}(1|2)$ and $\tilde{B}(2|2)$ are also SNS matrices. Let B' be a nearly decomposable (0, 1) matrix of order $n - 1$ with $B' \leqslant B$. Then $B' * \tilde{B}$ is a conversion of B'. Since B' is fully indecomposable, $(B' * \tilde{B})(i|j)$ is an SNS matrix and

$$\det\left(B' * \tilde{B}\right)(i|j) \det \tilde{B}(i|j) > 0 \qquad \text{for} \quad 1 \leqslant i, j \leqslant 2.$$

Lemma 14 now implies that A is convertible if and only if

$$A * \left[\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & & \\ 1 & & & & & \\ 0 & & & B' & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right]$$

is convertible. By the minimality of A we conclude that $B = B'$ and hence that B is nearly decomposable. Since A has at least three 1's in each row, the total number of 1's contained in B is at least $3(n-1)-2$. Proposition 6 implies that $n-1 \leqslant 2$. We conclude that $n = 3$ and that $A = J_3$. The proof is now complete. ■

4. SEYMOUR AND THOMASSEN'S CHARACTERIZATION

We now give a new proof of Theorem 4. It proceeds in a manner similar to that of the previous proof; however, the crucial tool is Proposition 7 rather than Proposition 6.

Again, we begin with a technical lemma, which is the digraph version of Lemma 14. For a subset ϵ of arcs, we denote by $\mathscr{D}^{\epsilon}$ the digraph obtained from $\mathscr{D}$ by removing the arcs belonging to ϵ.

LEMMA 15. *Let $\mathscr{D}$ be a digraph, and suppose that there exists a vertex v such that the indegree and the outdegree of v both equal 2 and $\mathscr{D}_{\{v\}}$ is a strongly connected convertible digraph. Assume (v, a), (v, b), (c, v), and (d, v) are the arcs of $\mathscr{D}$ which meet the vertex v, where $a \neq b$ and $c \neq d$, and let* $\mathbf{wt}$ *be a conversion of $\mathscr{D}_{\{v\}}$. Then $\mathscr{D}$ is convertible if and only if*

(i) *the weight of any path from a to c in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ has the same parity as a constant $\mathbf{wt}_{a \rightsquigarrow c}$;*

(ii) *the weight of any path from a to d in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ has the same parity as a constant $\mathbf{wt}_{a \rightsquigarrow d}$;*

(iii) *the weight of any path from b to c in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ has the same parity as a constant $\mathbf{wt}_{b \rightsquigarrow c}$;*

(iv) *the weight of any path from b to d in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ has the same parity as a constant $\mathbf{wt}_{b \rightsquigarrow d}$; and*

(v) $\mathbf{wt}_{a \rightsquigarrow c} + \mathbf{wt}_{a \rightsquigarrow d} + \mathbf{wt}_{b \rightsquigarrow c} + \mathbf{wt}_{c \rightsquigarrow d} \equiv 0 \pmod 2$.

Proof. First suppose that $\mathscr{D}$ is a convertible digraph. It follows from Theorem 13 that there exists a conversion $\widetilde{\mathbf{wt}}$ of $\mathscr{D}$ which agrees with $\mathbf{wt}$ on the arcs of $\mathscr{D}_{\{v\}}$. Each path from a to c in $\mathscr{D}_{\{v\}}$ determines a directed cycle in $\mathscr{D}$. Namely, if $a = c$ then the empty path corresponds to the directed cycle $a \to c \to a$, and if $a \neq c$ then the path γ from a to c corresponds to the directed cycle $\gamma \to v \to a$. It follows that the weights of any pair of paths from a to c in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ have the same parity (the weight of the empty path equals 0). A similar argument holds for paths from a to d, b to c, and b to d, and we conclude that (i)–(iv) hold. Since $\widetilde{\mathbf{wt}}\,\gamma \equiv 1 \bmod 2$ for all directed cycles γ of $\mathscr{D}$, we have

$$\begin{aligned}
\mathbf{wt}_{a\rightsquigarrow c} &\equiv \widetilde{\mathbf{wt}}(c, v) + \widetilde{\mathbf{wt}}(v, a) + 1 \pmod 2,\\
\mathbf{wt}_{a\rightsquigarrow d} &\equiv \widetilde{\mathbf{wt}}(d, v) + \widetilde{\mathbf{wt}}(v, a) + 1 \pmod 2,\\
\mathbf{wt}_{b\rightsquigarrow c} &\equiv \widetilde{\mathbf{wt}}(c, v) + \widetilde{\mathbf{wt}}(v, b) + 1 \pmod 2,\\
\mathbf{wt}_{b\rightsquigarrow d} &\equiv \widetilde{\mathbf{wt}}(d, v) + \widetilde{\mathbf{wt}}(v, b) + 1 \pmod 2.
\end{aligned} \tag{8}$$

By summing the equations in (8) we conclude that statement (v) holds.

Now suppose that statements (i)–(v) hold. Then the system of linear equations (8) with unknowns $\widetilde{\mathbf{wt}}(v, a)$, $\widetilde{\mathbf{wt}}(v, b)$, $\widetilde{\mathbf{wt}}(c, v)$, and $\widetilde{\mathbf{wt}}(d, v)$ has a solution. It follows that the weighting $\widetilde{\mathbf{wt}}$ of $\mathscr{D}$ which agrees with $\mathbf{wt}$ on the arcs of $\mathscr{D}_{\{v\}}$ and agrees with a solution to (8) on the arcs (c, v), (d, v), (v, a), and (v, b) is a conversion of $\mathscr{D}$. ■

We conclude with a proof of Theorem 4. The basic outline for the proof that an even digraph $\mathscr{D}$ contains a splitting of an odd-double cycle is as follows. The proof is by induction on the number of vertices of $\mathscr{D}$.

(1) We may assume that $\mathscr{D}$ is strongly connected and that $\mathscr{D}$ is minimal in the sense that removing any arc of $\mathscr{D}$ results in a convertible digraph or a digraph that is not strongly connected.

(2) If $\mathscr{D}$ has a vertex with indegree or outdegree 1, then $\mathscr{D}$ is a splitting of a digraph E, and we apply induction to E.

(3) Assume that the indegree and the outdegree of each vertex of $\mathscr{D}$ are each at least 2. By Proposition 8, there exists a vertex v such that $\mathscr{D}_v$ is strongly connected.

(4) Argue that the minimality of $\mathscr{D}$ implies that v has indegree equal to 2 and outdegree equal to 2.

(5) Use the minimality of $\mathscr{D}$ and Lemma 15 to show that $\mathscr{D}_v$ is a minimal strong digraph.

(6) Use Proposition 7 to conclude that $\mathscr{D}$ is an odd-double cycle.

Proof of Theorem 4. It is easy to verify that every weighting of a splitting of a k-double cycle, where $k \geqslant 3$ is an odd integer, has a directed cycle of even weight. Thus a digraph $\mathscr{D}$ is even if it contains a splitting of a k-double cycle for some odd integer $k \geqslant 3$.

Conversely, let $\mathscr{D}$ be an even digraph. We prove that $\mathscr{D}$ contains a splitting of an odd-double cycle by induction on the number of vertices. If $\mathscr{D}$ is not strongly connected, then we may apply induction to the strongly connected components of $\mathscr{D}$. Thus we may assume that $\mathscr{D}$ is strongly connected and that $\mathscr{D}$ is minimal in the sense that removing any arc of $\mathscr{D}$ results in a digraph which either is convertible or is not strongly connected.

Suppose there exists a vertex of indegree or outdegree 1; then $\mathscr{D}$ is a splitting of an even digraph with fewer vertices. It follows by induction that $\mathscr{D}$ contains a splitting of an odd-double cycle. Thus we may assume that every vertex of $\mathscr{D}$ has indegree and outdegree at least 2. By Proposition 8, there exists a vertex v such that $\mathscr{D}_{\{v\}}$ is strongly connected. If $\mathscr{D}_{\{v\}}$ is even, then by induction $\mathscr{D}_{\{v\}}$, and hence $\mathscr{D}$, contains a splitting of an odd-double cycle. Thus we may also assume that $\mathscr{D}_{\{v\}}$ is convertible. Let $\mathbf{wt}$ be a conversion of $\mathscr{D}_{\{v\}}$.

Suppose that the outdegree of v is at least 3, and let (v, x), (v, y), and (v,z) be three arcs of $\mathscr{D}$. Then $\mathscr{D}^{\{(v,y)\}}$ is strongly connected and, by the minimality of $\mathscr{D}$, is also convertible. Let $\mathbf{wt}_y$ be a conversion of $\mathscr{D}^{\{(v,y)\}}$. Similarly, $\mathscr{D}^{\{(v,z)\}}$ is strongly connected and has a conversion $\mathbf{wt}_z$. Since $\mathscr{D}^{\{(v,y)(v,z)\}}$ is strongly connected, Theorem 13 implies that we may choose $\mathbf{wt}_z$ so that it agrees with $\mathbf{wt}_y$ on the arcs of $\mathscr{D}^{\{(v,y),(v,z)\}}$. Let $\mathbf{wt}$ be the weighting of $\mathscr{D}$ which agrees with $\mathbf{wt}_y$ on the arcs of $\mathscr{D}^{\{(v,y)\}}$ and with $\mathbf{wt}_z$ on the arcs of $\mathscr{D}^{\{(v,z)\}}$. Then it is easy to verify that every directed cycle γ of $\mathscr{D}$ satisfies $\mathbf{wt}\,\gamma \equiv 1 \bmod 2$, contrary to our assumption that $\mathscr{D}$ is an even digraph. Thus v has outdegree 2. A similar argument shows that v has indegree 2.

Let a and b be the vertices such that (v, a) and (v, b) are arcs in $\mathscr{D}$, and let c and d be the vertices such that (c, v) and (d, v) are arcs in $\mathscr{D}$. By the minimality of $\mathscr{D}$ and Theorems 12 and 13, there exists a conversion $\mathbf{wt}'$ of $\mathscr{D}^{\{(v,a)\}}$ which agrees with $\mathbf{wt}$ on the arcs of $\mathscr{D}_{\{v\}}$. It follows that the weights of any two paths from vertex b to vertex c in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ have the same parity, say $\mathbf{wt}_{b \rightsquigarrow c}$. Similarly the weights of any two paths from vertex b to vertex d in $(\mathscr{D}_{\{v\}}, \mathbf{wt})$ have the same parity, say $\mathbf{wt}_{b \rightsquigarrow d}$. By considering $\mathscr{D}^{\{(v,b)\}}$, the constants $\mathbf{wt}_{a \rightsquigarrow c}$ and $\mathbf{wt}_{a \rightsquigarrow d}$ are also defined. Suppose there exists an arc α of $\mathscr{D}_{\{v\}}$ such that the digraph $\mathscr{D}_{\{v\}}^{\alpha}$ obtained from $\mathscr{D}_{\{v\}}$ by removing α is strongly connected. Then the digraph $\mathscr{D}^{\{\alpha\}}$ obtained by removing α from $\mathscr{D}$ is also strongly connected. By the minimality of $\mathscr{D}$ and Theorem 13, there exists a conversion $\mathbf{wt}^*$ of $\mathscr{D}^{\{\alpha\}}$ which agrees with $\mathbf{wt}$ on the arcs of $\mathscr{D}_{\{v\}}^{\alpha}$. Thus by Theorem 13 each path from vertex a to vertex c in $\mathscr{D}_{\{v\}}^{\alpha}$ has the same weight, say $\mathbf{wt}^*_{a \rightsquigarrow c}$. Similar statements hold for paths from a to d, b to c, and b to d. By Lemma 15 we have

$$\mathbf{wt}^*_{a \rightsquigarrow c} + \mathbf{wt}^*_{a \rightsquigarrow d} + \mathbf{wt}^*_{b \rightsquigarrow c} + \mathbf{wt}^*_{b \rightsquigarrow d} \equiv 0 \pmod 2. \tag{9}$$

But

$$\mathbf{wt}^*_{a \rightsquigarrow c} \equiv \mathbf{wt}_{a \rightsquigarrow c} \pmod 2,$$

$$\mathbf{wt}^*_{a \rightsquigarrow d} \equiv \mathbf{wt}_{a \rightsquigarrow d} \pmod 2,$$

$$\mathbf{wt}^*_{b \rightsquigarrow c} \equiv \mathbf{wt}_{b \rightsquigarrow c} \pmod 2,$$

$$\mathbf{wt}^*_{b \rightsquigarrow d} \equiv \mathbf{wt}_{b \rightsquigarrow d} \pmod 2,$$

and hence Lemma 15 implies that $\mathscr{D}$ is convertible, contrary to our assumption. We conclude that $\mathscr{D}_{\{v\}}$ is a minimal strong digraph. Since the indegree and outdegree of each vertex of $\mathscr{D}$ are at least 2, at most two vertices of $\mathscr{D}_{\{v\}}$ have indegree less than 2 and at most two vertices of $\mathscr{D}_{\{v\}}$ have outdegree less than 2. Thus $\mathscr{D}_{\{v\}}$ has at least $2(n-2)$ arcs. It follows from Proposition 7 that $\mathscr{D}_{\{v\}}$ is a double tree and the tree has degree sequence $1, 1, 2, 2, \ldots, 2$. Thus $\mathscr{D}_{\{v\}}$ is a double path. By considering the indegree and outdegree of vertices in $\mathscr{D}$ we conclude that $\mathscr{D}$ is a double cycle. Since an even double cycle is an even digraph, n is odd. The proof is now complete. ■

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